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A note on boundary-layer growth at an axisymmetric rear stagnation point

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This paper presents the analysis and numerical solution for the axisymmetric analogue of the problem considered by Proudman & Johnson (1962) and Robins & Howarth (1972).

1. Introduction

The purpose of this paper is to extend the work of Proudman & Johnson (1962) and Robins & Howarth (1972) on boundary-layer growth at a two-dimensional rear stagnation point to the axisymmetric problem (e.g. the rear of a sphere). The analysis follows closely (at least in principle) that of the above two papers, where it is set out in detail; consequently the exposition here will be kept as short as possible.

2. Equations of motion and similarity solution

With a non-dimensional stream function F, non-dimensional boundary-layer normal co-ordinate y and non-dimensional time t, the equations of motion and the boundary conditions give the following problem for F, corresponding to equations (2.5) and (2.6) in Robins & Howarth:

$$F_{yyy} - 2FF_{yy} - 1 + F_y^2 = F_{yt}, \tag{2.1}$$

together with

$$\left.\begin{aligned} F = F_y = 0 \quad \text{on} \quad y = 0 \quad \text{for} \quad t \neq 0, \\ F_y \to 1 \quad \text{as} \quad y \to \infty, \\ F_y = 1 \quad \text{at} \quad t = 0 \quad \text{for} \quad y \neq 0. \end{aligned}\right\} \tag{2.2}$$

The factor of 2 in (2.1) is the only difference at this stage between the two-dimensional and axisymmetric cases. Following Proudman & Johnson, we seek a similarity solution of the inviscid form of (2.1), and also assume exponential decay of vorticity as $y \to \infty$. It turns out that the similarity solution is of the form

$$F(y,t) = e^{2t} f(y\,e^{-2t}). \tag{2.3}$$

Hence the relevant outer variable η is given by $\eta = e^{-2t}$, and we obtain the stream function in the outer region, say $F(y,t) = G(\eta, t)$, satisfying

$$e^{-6t} G_{\eta\eta\eta} - 2e^{-4t} GG_{\eta\eta} - 1 + e^{-4t} G_\eta^2 = e^{-2t}(G_{\eta t} - 2G_\eta - 2\eta G_{\eta\eta}), \tag{2.4}$$

with

$$G_\eta(\eta, t) \to e^{2t} \quad \text{as} \quad \eta \to \infty. \tag{2.5}$$

The inner co-ordinate will be y itself, so we formally write $F(y,t) = g(y,t)$ in this region and obtain the inner equation

$$g_{yyy} - 2gg_{yy} - 1 + g_y^2 = g_{yt}, \tag{2.6}$$

with boundary conditions

$$g = g_y = 0 \quad \text{on} \quad y = 0. \tag{2.7}$$

3. The analytical solution

Following a technique identical to that of Robins & Howarth, the outer expansion is found to be

$$G(\eta,t) = e^{2t}F_0(\eta) + F_1(\eta) + (e^{-t}/A)\,F_2(\eta), \tag{3.1}$$

where F_0 satisfies

$$2(F_0 - \eta)\,F_0'' + 1 - F_0'^2 = 0,\;\; F_0(0) = 0,\;\; F_0'(\eta) \to 1 \text{ exponentially as } \eta \to \infty, \tag{3.2}$$

with solution

$$\eta = -2A(\eta - F_0)^{\frac{1}{2}} - 4A^2 \log\,[1 - (\eta - F_0)^{\frac{1}{2}}/2A]. \tag{3.3}$$

This is the axisymmetric similarity solution, first obtained by Johnson in her Ph.D. thesis, though not published elsewhere. The constant A corresponds to the constant c in the two-dimensional case, and again represents an uncertainty in the precise location of the time origin. The functions F_1 and F_2 satisfy

$$(F_0 - \eta)\,F_1'' - (1 + F_0')\,F_1' + F_0''F_1 = 0, \quad F_1'(\infty) = 0, \tag{3.4}$$

with solution

$$F_1 = A_1 Y^2 + A_1'(1 - Y), \tag{3.5}$$

where $Y = 1 + F_0'$, and

$$2(F_0 - \eta)\,F_2'' - (2F_0' + 3)\,F_2' + 2F_0''F_2 = 0, \quad F_2'(\infty) = 0, \tag{3.6}$$

with solution

$$F_2 = A_2(2Y - 1) + A_2'\{Y^{\frac{1}{2}}(1 + Y)\,(2 - Y)^2 + 2(2Y - 1)\sin^{-1}(\tfrac{1}{2}Y)^{\frac{1}{2}}\}. \tag{3.7}$$

Here A_1, A_1', A_2 and A_2' are constants to be determined by matching with the inner solution.

Similarly, the inner expansion is found to be

$$f(y,t) = f_0(y) + \frac{e^{-t}}{A} f_1(y) + \frac{e^{-2t}}{A^2} f_2(y) + \frac{e^{-\frac{5}{2}t}}{A^{\frac{5}{2}}} f_{\frac{5}{2}}(y), \tag{3.8}$$

where

$$f_0''' - 2f_0'' - 1 + f_0'^2 = 0, \quad f_0(0) = f_0'(0) = 0 \tag{3.9}$$

(this is the equation for the solution for a forward axisymmetric stagnation point, with a change in sign in f_0; it will be seen that in order to match with the outer solution, the correct asymptotic form will be

$$f_0 \sim -y + \delta + \exp, \tag{3.9a}$$

obtained by choosing $f_0''(0) = -1\cdot 3120$ in its numerical integration). We then have

$$f_1''' - 2f_0 f_1'' + (2f_0' + 1)f_1' - 2f_0'' f_1 = 0, \quad f_1(0) = f_1'(0) = 0, \tag{3.10}$$

$$f_2''' - 2f_0 f_2'' + 2(f_0' + 1)f_2' - 2f_0'' f_2 = 2f_1 f_1'' - f_1'^2, \quad f_2(0) = f_2'(0) = 0, \tag{3.11}$$

$$f_{\frac{5}{2}}''' - 2f_0 f_{\frac{5}{2}}'' + (2f_0' + \tfrac{5}{2})f_{\frac{5}{2}}' - 2f_0'' f_{\frac{5}{2}} = 0, \quad f_{\frac{5}{2}}(0) = f_{\frac{5}{2}}'(0) = 0. \tag{3.12}$$

The asymptotic forms of these functions are

$$f_1 \sim \alpha_1\{\tfrac{2}{3}y^{\frac{3}{2}} - \delta y^{\frac{1}{2}} + (\tfrac{1}{4}\delta^2 + \tfrac{1}{8})\, y^{-\frac{1}{2}} + O(y^{-\frac{3}{2}})\} + \alpha_1' + \exp, \tag{3.13}$$

$$f_2 \sim -\tfrac{1}{6}y^2 + (\alpha_2 + \tfrac{1}{3}\delta)\, y - 2^{\frac{3}{2}}\alpha_1' y^{\frac{1}{2}} - \tfrac{3}{8}\log y + (\alpha_2' - \alpha_2\delta - \tfrac{1}{6}\delta^2) + O(y^{-\frac{1}{2}}) + \exp, \tag{3.14}$$

$$f_{\frac{5}{2}} \sim \alpha_{\frac{5}{2}}(\tfrac{4}{3}y^{\frac{3}{4}} - \delta y^{-\frac{1}{4}} + O(y^{-\frac{5}{4}})) + \alpha_{\frac{5}{2}}' + \exp, \tag{3.15}$$

for some constants α_1, α_1', α_2, α_2', $\alpha_{\frac{5}{2}}$ and $\alpha_{\frac{5}{2}}'$.

Although these solutions were originally obtained in the usual step-by-step manner, they are presented together for brevity. For the same reason we shall here do the matching all at once. Expanding the outer solution for small η, and rewriting in inner variables, we obtain

$$\begin{aligned} G(\eta, t) \sim & -y + A_1' + \left(\frac{2^{\frac{3}{2}}}{3} y^{\frac{3}{2}} - 2^{\frac{1}{2}} A_1' y^{\frac{1}{2}} - A_2\right) \frac{e^{-t}}{A} \\ & + \{-\tfrac{1}{6}y^2 + (2A_1 + \tfrac{1}{3}A_1')\, y + 2^{\frac{3}{2}} A_2 y^{\frac{1}{2}}\} \frac{e^{-2t}}{A^2} + \tfrac{16}{3} A_2' y^{\frac{3}{4}} \frac{e^{-\frac{5}{2}t}}{A^{\frac{5}{2}}} + O(e^{-3t}). \end{aligned} \tag{3.16}$$

We see that $f_0 \sim -y$ as $y \to \infty$, as previously stated, hence $f_0''(0) = -1\cdot 3120$, and the constant δ is then $0\cdot 568\ldots$. We then need $A_1' = \delta$. Next $\alpha_1 = \sqrt{2}$. To satisfy this we need $f_1''(0) = 1\cdot 2072\ldots$, and α_1' is then $0\cdot 1205$. Then $A_2 = -\alpha_1' = -0\cdot 1205$. Also $\alpha_2 = 2A_1$ and $\alpha_{\frac{5}{2}} = 4A_2'$.

As in the two-dimensional problem, we have some indeterminacy in that A_1 and A_2' are not determined. They are found by comparison with the numerical solution. Note finally that logarithmic terms will occur at the next stage of the expansion, and also that, as in the two-dimensional case, it can be shown that linearized eigenfunctions of the inner problem (cf. Kelly 1962) can be relegated to a point further on in the expansion.

4. Numerical solution and comparison with numerical results

The numerical integration of (2.1) and (2.2) was undertaken in exactly the same way as in the two-dimensional case, and the details will not be repeated here. The first task in the comparison of analytical and numerical results is to verify that (3.3) is the relevant first outer solution, as was done by Proudman & Johnson for the two-dimensional case. A first integral of (3.1) is

$$A(1 + F_0') = (\eta - F_0)^{\frac{1}{2}}. \tag{4.1}$$

A graph of $e^{-t}(y - F)^{\frac{1}{2}}$ against F_y does indeed give a straight line, except near $y = 0$, as expected, and the value of A that emerges is

$$A = 0\cdot 255. \tag{4.2}$$

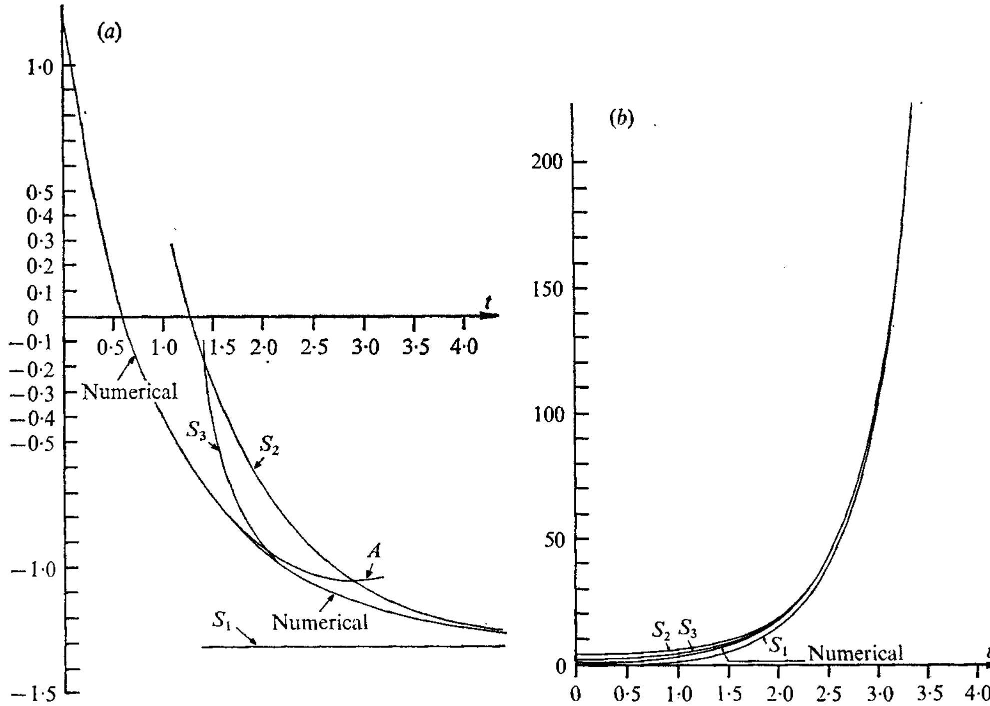

FIGURE 1. (a) Axisymmetric skin friction. (b) Axisymmetric displacement thickness.

Next, using the same procedure as in Robins & Howarth, we find that

$$A_1 = -0{\cdot}86, \quad A_2' = 0{\cdot}11. \tag{4.3}$$

Curves of analytical and numerical results for the non-dimensional skin friction $f_{yy}(0,t)$ and displacement thickness

$$\Delta = \int_0^\infty (1-F')\,dy$$

of the whole flow are presented in figures 1 (a) and (b). S_1, S_2 and S_3 denote respectively one-, two- and three-term approximations.

The value of t at which the skin friction first becomes negative was found to be $t = 0{\cdot}595$. Boltze's estimate for this time from the series for small t (see Schlichting 1960, p. 128) developed by Goldstein & Rosenhead (1936) and Squire (1954) was 0·589. As a matter of interest, the next two terms of this series were calculated by solving the associated differential equations numerically, and Shanks's transformation was then applied, twice. The resulting skin-friction curve is marked A. It will be seen that, taken together, the two series predict the skin friction over practically the whole of the time range. The large time series is valid back to about $t = 2{\cdot}25$, which is better than in the two-dimensional case; however, it should be borne in mind that once reverse flow has set in, the exponential growth rate is twice as fast as in the two-dimensional case, and so presumably the similarity solution emerges more quickly.

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REFERENCES

GOLDSTEIN, S. & ROSENHEAD, L. 1936 *Proc. Camb. Phil. Soc.* **32**, 392.

KELLY, R. E. 1962 *J. Fluid Mech.* **13**, 449.

PROUDMAN, I. & JOHNSON, K. 1962 *J. Fluid Mech.* **12**, 161.

ROBINS, A. J. & HOWARTH, J. A. 1972 *J. Fluid Mech.* **56**, 161.

SCHLICHTING, H. 1960 *Boundary Layer Theory*, 2nd edn. Pergamon.

SQUIRE, L. C. 1954 *Phil. Mag.* **45** (7), 1272.